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Unitary minimal models of $\mathcal{SW}(3/2, 3/2, 2)$ superconformal algebra and manifolds of G_2 holonomy

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Abstract: The $\mathcal{SW}(3/2, 3/2, 2)$ superconformal algebra is a $\mathcal{W}$ algebra with two free parameters. It consists of 3 superconformal currents of spins 3/2, 3/2 and 2. The algebra is proved to be the symmetry algebra of the coset $\frac{su(2)\oplus su(2)\oplus su(2)}{su(2)}$. At the central charge $c = 10\frac{1}{2}$ the algebra coincides with the superconformal algebra associated to manifolds of G_2 holonomy. The unitary minimal models of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra and their fusion structure are found. The spectrum of unitary representations of the G_2 holonomy algebra is obtained.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry, Integrable Field Theories.

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1. Introduction

Recently the manifolds of exceptional holonomy attracted much attention. These are 7-dimensional manifolds of G_2 holonomy and 8-dimensional manifolds of Spin(7) holonomy. They are considered in the context of the string theory compactifications.

The supersymmetric nonlinear sigma models on the manifolds of exceptional holonomy are described by conformal field theories, their superconformal chiral algebras were constructed in [1]. We will call them the G_2 and Spin(7) superconformal algebras. These are nonlinear $\mathcal{W}$–algebras ([2], for review see [3]) of central charge $10\frac{1}{2}$ and 12 respectively. The conformal field theories were further studied in [4]–[11].

The Spin(7) algebra is identified [4] with the $\mathcal{SW}(3/2,2)$ superconformal algebra [12]–[17], existing at generic values of the central charge. It consists of the $N=1$ superconformal algebra extended by its spin-2 superprimary field. The unitary representation theory of the $\mathcal{SW}(3/2,2)$ algebra is studied in [5], where complete list of unitary representations is determined (including the $c=12$ model, corresponding to the Spin(7) manifolds).

In this paper we identify the G_2 algebra with the $\mathcal{SW}(3/2,3/2,2)$ superconformal algebra (in notations of [3]) at the central charge $c=10\frac{1}{2}$ and the coupling constant (see below) $\lambda=0$. The $\mathcal{SW}(3/2,3/2,2)$ algebra was first constructed in [14] (see also [15]). It is superconformal $\mathcal{W}$-algebra, which besides the energy-momentum supercurrent (the first "3/2" in $\mathcal{SW}(3/2,3/2,2)$) contains two supercurrents of spins 3/2 and 2. The $\mathcal{SW}(3/2,3/2,2)$ algebra has two generic parameters. Along with the central charge there is a free coupling λ (the self-coupling of the spin-3/2 superprimary field), which is not fixed by Jacobi identities.

In [17] the $\mathcal{SW}(3/2,3/2,2)$ algebra is shown to be the symmetry algebra of the quantized Toda theory corresponding to the $D(2|1;\alpha)$ Lie superalgebra (the only simple Lie superalgebra with free parameter). In the same ref. [17] the free field representation of the $\mathcal{SW}(3/2,3/2,2)$ algebra is constructed.

We study different aspects of the $\mathcal{SW}(3/2,3/2,2)$ algebra in the present paper. First we find that the $\mathcal{SW}(3/2,3/2,2)$ algebra is the symmetry algebra of the diagonal coset

$$\frac{su(2)_{k_1}\oplus su(2)_{k_2}\oplus su(2)_2}{su(2)_{k_1+k_2+2}}\,. \tag{1.1}$$

We define highest weight representations of the algebra and study their unitarity. The unitary minimal models are described by the coset (1.1). Their central charge and coupling λ are given by

$$c_{k_1,k_2} = \frac{9}{2}+\frac{6}{k_1+k_2+4}-\frac{6}{k_1+2}-\frac{6}{k_2+2}\,, \tag{1.2}$$

$$\lambda_{k_1,k_2} = \frac{4\sqrt{2}\,(k_1-k_2)\,(2\,k_1+k_2+6)\,(k_1+2\,k_2+6)}{3\,(3\,k_1\,k_2\,(k_1+k_2+6))^{1/2}\,(k_1+2)\,(k_2+2)\,(k_1+k_2+4)}. \tag{1.3}$$

We also obtain all the values of c and λ, where the $\mathcal{SW}(3/2,3/2,2)$ algebra has continuous spectrum of unitary representations. One such model ($c=10\frac{1}{2}$, $\lambda=0$), which corresponds to the G_2 algebra, is discussed in details, the full spectrum of unitary representations is obtained. We also present the complete list of the minimal model representations and their fusion rules.

The diagonal coset constructions of type $(g \oplus g)/g$ were found very useful in the description of minimal models of different conformal algebras. The minimal models of the Virasoro algebra [18] ($c_k = 1 - \frac{6}{(k+2)(k+3)}$) correspond to the diagonal coset construction [19]

$$\frac{su(2)_k \oplus su(2)_1}{su(2)_{k+1}}, \qquad k \in \mathbb{N}. \tag{1.4}$$

The coset (B.4) is found [20] to form the minimal models of the $N = 1$ superconformal algebra ([21, 22, 23] and appendix B). The minimal models of the $\mathcal{W}_N$ algebra [24] are the $su(N)$ diagonal cosets

$$\frac{su(N)_k \oplus su(N)_1}{su(N)_{k+1}}, \qquad k \in \mathbb{N}. \tag{1.5}$$

We present here the first example (to our knowledge) of the conformal chiral algebra, corresponding to the diagonal coset of type $\frac{g \oplus g \oplus g}{g}$. It is nontrivial fact that the coset space (1.1) has the same symmetry algebra for different k_1 and k_2. It can be explained, probably, by the connection of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra to the Lie superalgebra $D(2|1;\alpha)$, which has a free parameter unlike the other simple Lie algebras.

The $\mathcal{SW}(3/2, 3/2, 2)$ algebra contains two fields of spin 3/2 and three fields of spin 2, making enough room for embedding of different subalgebras, such as the $N=0$ (Virasoro) and the $N=1$ conformal algebras. Besides the trivial $N=1$ subalgebra (generated by the super energy-momentum tensor) there are 3 different $N=1$ superconformal subalgebras of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra. These embeddings play a crucial role in the understanding of the representation theory of the algebra.

There are four types of highest weight representations of the algebra: Neveu-Schwarz (NS), Ramond and two twisted sectors. (The twisted sectors are defined only in the case of vanishing coupling λ.)

The minimal models are labeled by two natural numbers: k_1 and k_2. The NS and Ramond minimal model representations can be arranged in the form of 3-dimensional table, similarly to the 2-dimensional tables of representations of the $N=0$ and the $N=1$ conformal algebras. The fusion rules also satisfy the "$su(2)$ pattern" of the $N = 0$ and $N=1$ minimal model fusions.

The set of the G_2 algebra representations consists of 4 sectors: NS, Ramond and two twisted. There are continuous spectrum representations in every sector. We prove, that the G_2 conformal algebra is the extended version of the $\mathcal{SW}(3/2, 2)$ algebra at $c = 10\frac{1}{2}$. Due to this fact we get all the G_2 unitary representations from the known spectrum [5] of the $\mathcal{SW}(3/2, 2)$ algebra.

The paper is organized as follows. After reviewing the structure of $\mathcal{SW}(3/2, 3/2, 2)$ in section 2 we prove in section 3 that the algebra is the symmetry algebra of the coset space (1.1). In section 4 we discuss different embeddings of the $N = 1$ superconformal algebra into the $\mathcal{SW}(3/2, 3/2, 2)$ algebra and obtain the unitarity restrictions on the values of c and λ. In section 5 the highest weight representations of the algebra are introduced, the zero mode algebras in different sectors are discussed. In section 6 we concern with the minimal models of the algebra: we explain how the spectrum of unitary representations is obtained, discuss the fusion rules, and give two examples of the $\mathcal{SW}(3/2,3/2,2)$ minimal models in terms of the $N=2$ superconformal minimal models. Section 7 is devoted to the G_2 algebra, the superconformal algebra associated to the manifolds of G_2 holonomy. We find it convenient to put some useful (but in some cases lengthy) information in the closed form in the five appendices.

We have to note that substantial part of the calculations was done with a help of *Mathematica* package [25] for symbolic computation of operator product expansions.

2. Structure of the $\mathcal{SW}(3/2,3/2,2)$ algebra

Here we review the structure of $\mathcal{SW}(3/2,3/2,2)$ algebra, which was first constructed in [14].

The $\mathcal{SW}(3/2,3/2,2)$ algebra is an extension of $N=1$ superconformal algebra by two superconformal multiplets of dimensions 3/2 and 2.

A superconformal multiplet $\widehat{\Phi}=(\Phi,\Psi)$ of dimension Δ consists of two Virasoro primary fields of dimensions Δ and $\Delta+1/2$. Under the action of the supersymmetry generator G they transform as

$$G(z)\,\Phi(w) = \frac{\Psi(w)}{z-w}\,, \tag{2.1}$$

$$G(z)\,\Psi(w) = \frac{2\,\Delta\,\Phi(w)}{(z-w)^2}+\frac{\partial\Phi(w)}{z-w}\,. \tag{2.2}$$

The $\mathcal{SW}(3/2,3/2,2)$ algebra multiplets are denoted by $I=(G,T)$, $\widehat{H}=(H,M)$ $(\Delta=\frac{3}{2})$, $\widehat{W}=(W,U)$ $(\Delta=2)$. The structure of the algebra is schematically given by

$$\begin{aligned}
\widehat{H}\times\widehat{H} &= I+\lambda\,\widehat{H}+\mu\,\widehat{W}\,,\\
\widehat{H}\times\widehat{W} &= \mu\,\widehat{H}+\lambda\,\widehat{W}\,,\\
\widehat{W}\times\widehat{W} &= I+\lambda\,\widehat{H}+\mu\,\widehat{W}+{:}\widehat{H}\widehat{H}{:}\,,
\end{aligned} \tag{2.3}$$

where

$$\mu=\sqrt{\frac{9\,c\,(4+\lambda^2)}{2\,(27-2\,c)}}\,, \tag{2.4}$$

and the c dependence of the coefficients is omitted.

The explicit operator product expansions (OPEs) are fixed by the fusions (2.3) and by $N=1$ superconformal invariance. We reproduce the OPEs in appendix A.

Unitarity is introduced by the standard conjugation relation $\mathcal{O}_n^\dagger=\mathcal{O}_{-n}$ for any generator except U. The commutation relation $[G_n,W_m]=U_{n+m}$, following from (2.1), requires the conjugation $U_n^\dagger=-U_{-n}$.

3. Coset construction

3.1 Preliminary discussion

We start from a few supporting arguments, that the coset theory (1.1) indeed possesses the $\mathcal{SW}(3/2,3/2,2)$ superconformal symmetry.

First, we note that formally $su(2)_2 \approx so(3)_1$ and the coset (1.1) can be written in the form of Kazama-Suzuki coset [26] $\frac{g \oplus so(\dim g - \dim h)_1}{h}$, where $g = su(2) \oplus su(2)$ and h is its $su(2)$ diagonal subalgebra. It means, that the chiral algebra contains $N=1$ superconformal algebra, obtained from the affine currents by a superconformal generalization of Sugawara construction (see ref. [26] for details).

All other currents, that constitute the chiral algebra should come in pairs of superpartners with difference of scaling dimensions equal to 1/2.

The central charge (1.2) of the coset models (1.1) has limiting point (when $k_1, k_2 \to \infty$) $c = 9/2$. All the known examples of minimal series have limiting central charge $c = n_B + \frac{1}{2} n_F$, where n_B and n_F are the number of bosonic and fermionic fields in the correspondent chiral algebra. Adopting the argument to our case we get, that the chiral algebra consists of three supercurrents (including the super-Virasoro operator).

The next argument follows from the simple observation [27], that if there is a sequence of subalgebra inclusions

$$g \supset h_1 \supset \cdots \supset h_n \tag{3.1}$$

then the coset theory can be decomposed to the tensor product

$$\frac{g}{h_n} = \frac{g}{h_1} \oplus \frac{h_1}{h_2} \oplus \ldots \oplus \frac{h_{n-1}}{h_n}\,. \tag{3.2}$$

In the case of coset (1.1) the inclusion sequence is

$$su(2)_{k_1} \oplus su(2)_{k_2} \oplus su(2)_2 \supset h_1 \supset su(2)_{k_1+k_2+2} \tag{3.3}$$

with 3 different choices of h_1: $su(2)_{k_1+2} \oplus su(2)_{k_2}$, $su(2)_{k_2+2} \oplus su(2)_{k_1}$, $su(2)_{k_1+k_2} \oplus su(2)_2$. The correspondent decompositions are:

$$\frac{su(2)_{k_1} \oplus su(2)_2}{su(2)_{k_1+2}} \oplus \frac{su(2)_{k_1+2} \oplus su(2)_{k_2}}{su(2)_{k_1+k_2+2}}\,, \tag{3.4}$$

$$\frac{su(2)_{k_2} \oplus su(2)_2}{su(2)_{k_2+2}} \oplus \frac{su(2)_{k_2+2} \oplus su(2)_{k_1}}{su(2)_{k_1+k_2+2}}\,, \tag{3.5}$$

$$\frac{su(2)_{k_1} \oplus su(2)_{k_2}}{su(2)_{k_1+k_2}} \oplus \frac{su(2)_{k_1+k_2} \oplus su(2)_2}{su(2)_{k_1+k_2+2}}\,. \tag{3.6}$$

All three contain coset spaces of type (B.4) with $k = k_1$, $k = k_2$ and $k = k_1 + k_2$ respectively. Therefore the chiral algebra contain 3 different $N = 1$ superconformal subalgebras (not including the trivial one, generated by the super energy-momentum tensor). This is possible if there are at least 3 operators of scaling dimension 2. They have 3 superpartners of dimensions 3/2 or 5/2. One field of dimension 3/2 can not serve as a superconformal generator for 3 different superconformal subalgebras. The case, when all three are of dimension 3/2 is also excluded, because then the algebra is trivially a sum of 3 commuting $N=1$ superconformal algebras.

Collecting the arguments we get that the only possibility that the chiral algebra consists of 6 fields of dimensions 3/2, 3/2, 2, 2, 2, 5/2, which can be combined to three supercurrents of dimensions 3/2, 3/2, 2, giving the $\mathcal{SW}(3/2, 3/2, 2)$ algebra.

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3.2 Explicit construction

In this section we prove by explicit construction that the coset (1.1) contains the $\mathcal{SW}(3/2, 3/2, 2)$ algebra. The method we use was first proposed in [20] for coset (B.4).

The $su(2)$ affine algebra is generated by 3 currents J_i, $i = 1, 2, 3$:

$$J_i(z)\, J_i(w) = \frac{k/2}{(z-w)^2} + \frac{\mathrm{i}\, \epsilon_{ijk}\, J_k(w)}{z-w}\,. \tag{3.7}$$

The g algebra consists of 3 commuting copies of the $su(2)$ algebra at levels k_1, k_2 and 2. The $su(2)$ on level 2 is realized by free fermions in the adjoint representation of $su(2)$:

$$\psi_i(z)\, \psi_j(w) = \frac{\delta_{ij}}{z-w}\,, \qquad i, j = 1, 2, 3\,. \tag{3.8}$$

Then the affine currents are expressed as

$$J_i = -\frac{\mathrm{i}}{2}\, \epsilon_{ijk}\, {:}\psi_j \psi_k{:}\,. \tag{3.9}$$

The affine algebra $h = su(2)_{k_1+k_2+2}$ is diagonally embedded in $g = su(2)_{k_1} \oplus su(2)_{k_2} \oplus su(2)_2$:

$$J_i^{(h)} = J_i^{(1)} + J_i^{(2)} + J_i^{(3)}\,. \tag{3.10}$$

The coset space g/h contains operators, constructed from the g currents, which commute with the currents of h. The energy-momentum tensor of the coset g/h is given by the Sugawara construction:

$$T = T^{(g)} - T^{(h)} = T^{(1)} + T^{(2)} + T^{(3)} - T^{(h)}\,, \tag{3.11}$$

where

$$T^{(n)} = \frac{1}{k_n + 2} \sum_{i=1}^{3} {:}J_i^{(n)} J_i^{(n)}{:}\,. \tag{3.12}$$

The general dimension-3/2 operator can be written as

$$\mathcal{O}_{3/2} = b_1 \sum_{i=1}^{3} {:}J_i^{(1)} \psi_i{:} + b_2 \sum_{i=1}^{3} {:}J_i^{(2)} \psi_i{:} + \mathrm{i}\, b_3\, {:}\psi_1 \psi_2 \psi_3{:}\,. \tag{3.13}$$

It should commute with the $J^{(h)}$ currents. This requirement leads to condition

$$b_3 = \frac{1}{2}(b_1\, k_1 + b_2\, k_2)\,. \tag{3.14}$$

One has two independent dimension-3/2 operators (G and H) in the coset theory, since there are two free parameters b_1 and b_2.

In order to close the algebra one needs 3 more operators M, W, U of scaling dimensions 2, 2, 5/2. The coset construction of all the 6 operators is given in appendix C.

We have explicitly checked, that the set of 6 operators T, G, H, M, W, U satisfy the OPEs of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra with central charge c and coupling λ given by (1.3).

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4. $N=1$ superconformal subalgebras

We start this section by observation, that 3 bosonic operators T, M, W do not generate a closed subalgebra because of $:GH:$ term in the OPE of M with W (A.5).

We will construct in this section different $N=1$ superconformal subalgebras of $\mathcal{SW}(3/2, 3/2, 2)$ and discuss the unitarity restrictions. The most general dimension-$\frac{3}{2}$ operator is[1]

$$\widetilde{G} = \alpha\, G + \beta\, H\,. \tag{4.1}$$

We calculate its OPE with itself:

$$\widetilde{G}(z)\,\widetilde{G}(w) = \frac{\frac{2}{3}\,c\,(\alpha^2+\beta^2)}{(z-w)^3} + \frac{2\,\widetilde{T}}{z-w}\,, \tag{4.2}$$

where

$$\widetilde{T} = (\alpha^2+\beta^2)\,T + \beta\,(\alpha + \frac{1}{2}\,\lambda\,\beta)\,M + \frac{2}{3}\,\mu\,\beta^2\,W\,, \tag{4.3}$$

and take the dimension-2 operator $\widetilde{T}$ as the Virasoro generator of the subalgebra. These two operators, $\widetilde{G}$ and $\widetilde{T}$, generate a closed subalgebra if the following two equations are satisfied:

$$\begin{aligned} 27\,\alpha^2 + 27\,\lambda\,\alpha\,\beta + (9\,\lambda^2 + 4\,\mu^2 + 9)\,\beta^2 - 9 &= 0\,, \\ \alpha^3 - \alpha + 3\,\alpha\,\beta^2 + \lambda\,\beta^3 &= 0\,. \end{aligned} \tag{4.4}$$

Then the central charge of the subalgebra is $\widetilde{c} = c\,(\alpha^2+\beta^2)$. Formally the lefthand side of the first equation should be multiplied by β. We removed it in order to exclude the trivial solution $\beta = 0, \alpha = 1$, corresponding to the obvious $N=1$ subalgebra.

The operator $T-\widetilde{T}$ generates another closed subalgebra, namely Virasoro algebra with central charge $c-\widetilde{c}$, and it is commutative with the $N=1$ superconformal subalgebra of $\widetilde{G}$ and $\widetilde{T}$.

The equations (4.4) are polynomial equations in α and β of orders 2 and 3. Generically there are 6 solutions. One should take into account the $\mathbb{Z}_2$ symmetry $\alpha \to -\alpha$ and $\beta \to -\beta$ of the equations, corresponding to the $\mathbb{Z}_2$ transformation of the $N=1$ superconformal algebra $\widetilde{G} \to -\widetilde{G}$, $\widetilde{T} \to \widetilde{T}$. So at any generic c and λ there are three $N=1$ superconformal subalgebras.

However, we are interested in real subalgebras, i.e. preserving the conjugation relation: the conjugation of the subalgebra $\widetilde{G}_n^\dagger = \widetilde{G}_{-n}$, $\widetilde{T}_n^\dagger = \widetilde{T}_{-n}$ should be consistent with the conjugation relations of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra. This is true, if α and β are real. Generically there can be 1 or 3 real solutions of (4.4).

Unitary representations of an algebra are necessarily unitary representations of all its real subalgebras. The $N=1$ superconformal algebra has unitary representations only at $\widetilde{c} \geq 3/2$ or at $\widetilde{c} = c_k^{N=1}$, $k \in \mathbb{N}$ (B.1). The central charges of all the real $N=1$ superconformal subalgebras should be from this set.

[1]The discussion applies to NS and Ramond sectors. One cannot mix G and H in the case of twisted sectors.

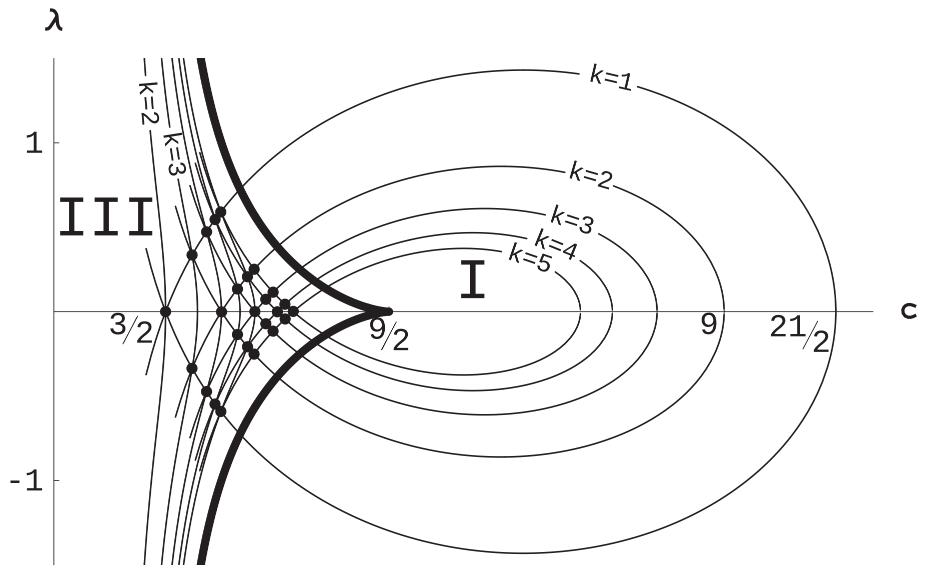

Figure 1: Unitary models of the $\mathcal{SW}(3/2,3/2,2)$ algebra.

We study the solutions of the set of equations (4.4) in the region $0 \le c < 27/2$. (At $c > 27/2$ the coupling μ becomes imaginary.) The results are presented in figure 1. The region $0 \le c < 27/2$ is divided to two parts by the curve (thick curve in figure 1)

$$4\,(9-2\,c)^3 = 243\,(2\,c-3)\,\lambda^2\,. \tag{4.5}$$

In the region I ($4\,(9-2\,c)^3 < 243\,(2\,c-3)\,\lambda^2$) there is one real solution of (4.4), in the region III ($4\,(9-2\,c)^3 > 243\,(2\,c-3)\,\lambda^2$) there are 3 real solutions.

The curves in figure 1 have constant subalgebra central charge along it: $\widetilde{c} = c_k^{N=1}$. We call them the unitary curves. Taking different $N=1$ subalgebras one gets different curves:

$$\lambda^2 = \frac{4\,(54-4\,c+9\,k-2\,c\,k)\,(4\,c-9\,k+2\,c\,k)^2}{243\,k\,(16\,c-18\,k+12\,c\,k-3\,k^2+2\,c\,k^2)} \tag{4.6}$$

$$\lambda^2 = \frac{4\,(9\,k-2\,c\,k-8\,c)\,(8\,c+2\,c\,k-9\,k-54)^2}{243\,(6+k)\,(16\,c-18\,k+12\,c\,k-3\,k^2+2\,c\,k^2)} \tag{4.7}$$

All the region under discussion is spanned by the curves (4.6), and (4.7). There are no real solutions of (4.4) corresponding to $\widetilde{c} > 3/2$.

In the region III the unitarity is restricted to the intersections of the unitary curves (the dots in figure 1). There are intersections of exactly 3 curves in every intersection point: two curves of type (4.6) with $k = k_1$ and $k = k_2$ and the third of type (4.7) with $k = k_1 + k_2$. The intersection points are given by (1.2), (1.3). The formula (1.2) is exactly the formula for central charge of coset theories (1.1). The central charges of three real subalgebras (we will call them the first, the second and the third $N=1$ subalgebras) at c and λ at the intersection point are $c_{k_1}^{N=1}$, $c_{k_2}^{N=1}$, $c_{k_1+k_2}^{N=1}$, they also coincide with the central charges of the three $N=1$ subalgebras of the coset (1.1) (see the decompositions (3.4), (3.5), (3.6)). We conclude that all unitary models of the $\mathcal{SW}(3/2,3/2,2)$ algebra in the region III are given by coset models (1.1). One can solve the equations (4.4) for the values of c and λ from (1.2), (1.3) to get the linear connection between the generators T, M, W and the Virasoro generators of the three subalgebras:

$$T = \frac{1}{2}\left(\widetilde{T}^{(1)}\,(k_1+4) + \widetilde{T}^{(2)}\,(k_2+4) - \widetilde{T}^{(3)}\,(k_1+k_2+2)\right), \tag{4.8}$$

$$\begin{aligned} M = {} & \frac{1}{(6\,k_1\,k_2\,(k_1+k_2+6))^{1/2}} \times \\ & \times \left(\widetilde{T}^{(3)}\,\frac{(k_1-k_2)(k_1+k_2+2)(k_1\,k_2-2\,k_1-2\,k_2-12)}{(k_1+2)(k_2+2)} - \right. \\ & \left. - \left(\widetilde{T}^{(1)}\,\frac{(k_1+4)(k_1+2\,k_2+6)(k_2^2+k_1\,k_2-2\,k_1+6\,k_2)}{(k_2+2)(k_1+k_2+4)} - (1\leftrightarrow 2)\right)\right), \end{aligned} \tag{4.9}$$

$$\begin{aligned} W = {} & \left(\frac{3\,{k_1}^2\,k_2+3\,k_1\,{k_2}^2+2\,{k_1}^2+2\,{k_2}^2+20\,k_1\,k_2+12\,k_1+12\,k_2}{24\,k_1\,k_2\,(k_1+k_2+6)}\right)^{1/2} \times \\ & \times \left(\widetilde{T}^{(3)}\,\frac{(k_1+k_2+2)(k_1\,k_2+4\,k_1+4\,k_2+12)}{(k_1+2)(k_2+2)} - \right. \\ & \left. - \left(\widetilde{T}^{(1)}\,\frac{(k_1+4)(k_2^2+k_1\,k_2+4\,k_1+6\,k_2+12)}{(k_2+2)(k_1+k_2+4)} + (1\leftrightarrow 2)\right)\right). \end{aligned} \tag{4.10}$$

In the region I the unitarity is restricted to the unitary curves (4.6). The region I models with c and λ satisfying (4.6) are expected to have continuous spectrum of unitary representations.

On the separating curve (4.5) the unitarity is restricted to the limiting points of (1.2), (1.3) at one k fixed and another k taken to infinity. (And $c = 9/2$, $\lambda = 0$, when both $k_1, k_2 \to \infty$.)

The Virasoro subalgebras, generated by $T - \widetilde{T}$, give no new restrictions on unitarity of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra.

The point $c = 10\frac{1}{2}$, $\lambda = 0$, which corresponds to the conformal algebra on G_2 manifold, is in the region I and lies on the (4.6) curve with $k = 1$. It means that the algebra have one real $N=1$ subalgebra and its central charge is $c_1^{N=1} = 7/10$ in agreement with results of ref. [1]. The generators of this real subalgebra are

$$\widetilde{G} = \left(\frac{27-2c}{3\,(9-2\,c)}\right)^{1/2} H\,, \tag{4.11}$$

$$\widetilde{T} = \frac{1}{3\,(9+2\,c)}\left((27-2\,c)\;T + 2\,(2\,c\;(27-2\,c))^{1/2}\,W\right). \tag{4.12}$$

(This is true for any $\lambda = 0$ model.)

The important question for understanding the structure of unitary representations of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra is how the algebra is decomposed to the representations of its real $N=1$ subalgebras? The decomposition is $\Phi_{11} + \Phi_{31}$ under subalgebras corresponding to the (4.6) curve and $\Phi_{11} + \Phi_{13}$ under subalgebras corresponding to the (4.7) curve, where Φ_{11} is the vacuum representation of the $N=1$ superconformal algebra, Φ_{13} and Φ_{31} are its degenerate representations, having null vector on level 3/2. The $\widetilde{T}$, $\widetilde{G}$ and T fields are in the Φ_{11} representation. Three other fields of $\mathcal{SW}(3/2,3/2,2)$ form Φ_{31} (or Φ_{13}) representation, they can be understood in this context as Φ_{31} (spin-3/2 field), $\widetilde{G}_{-1/2}\Phi_{31}$ (spin-2 field) and $\widetilde{T}_{-1}\Phi_{31}$ (spin-5/2 field). $\widetilde{T}_{-1}\widetilde{G}_{-1/2}\Phi_{31}$ is proportional to $\widetilde{G}_{-3/2}\Phi_{31}$ ($\approx$ $:\widetilde{G}\Phi_{31}:$), since there is a null state on level 3/2.

5. Highest weight representations

The $\mathcal{SW}(3/2,3/2,2)$ commutation relations admit two consistent choices of generator modes in the general case: NS and Ramond sectors; and two more then the coupling $\lambda = 0$: first twisted (tw1) and second twisted sectors (tw2).

NS sector. The modes of the bosonic operators (L_n, M_n, W_n) are integer ($n \in \mathbb{Z}$) and the modes of the fermionic operators (G_r, H_r, U_r) are half-integer ($r \in \mathbb{Z} + 1/2$).

Ramond sector. The modes of all the operators are integer.

First twisted sector. The modes of L_n, W_n, H_n operators are integer ($n \in \mathbb{Z}$) and the modes of G_r, M_r, U_r operators are half-integer ($r \in \mathbb{Z} + 1/2$).

Second twisted sector. The modes of L_n, G_n, W_n, U_n operators are integer ($n \in \mathbb{Z}$) and the modes of H_r, M_r operators are half-integer ($r \in \mathbb{Z} + 1/2$).

How can one understand the existence of four different sectors in terms of the coset construction (appendix C)? In order to get NS or Ramond sectors of the algebra one should take all the three fermions of $su(2)_2$ in NS or Ramond sectors respectively. The modes of $su(2)_{k_1}$ and $su(2)_{k_2}$ currents are integer. The twisted sectors ($k_1 = k_2$, since $\lambda = 0$) are obtained in less obvious way. First twisted sector: one takes one Ramond fermion (say, ψ_1) and two NS fermions (ψ_2 and ψ_3), the modes of $(J_1^{(1)} + J_1^{(2)})$, $(J_2^{(1)} - J_2^{(2)})$, $(J_3^{(1)} - J_3^{(2)})$ are integer and the modes of $(J_1^{(1)} - J_1^{(2)})$, $(J_2^{(1)} + J_2^{(2)})$, $(J_3^{(1)} + J_3^{(2)})$ are half-integer. Second twisted sector: one NS fermion (ψ_3) and two Ramond fermions (ψ_1 and ψ_2), the modes of $(J_1^{(1)} - J_1^{(2)})$, $(J_2^{(1)} - J_2^{(2)})$, $(J_3^{(1)} + J_3^{(2)})$ are integer and the modes of $(J_1^{(1)} + J_1^{(2)})$, $(J_2^{(1)} + J_2^{(2)})$, $(J_3^{(1)} - J_3^{(2)})$ are half-integer. One cannot define the modes of separate bosonic currents (e.g. $J_1^{(1)}$) in the twisted sectors.

The commutation relations of the $\mathcal{SW}(3/2,3/2,2)$ algebra include products of the generators. The formula (E.6) for the mode expansion of composite operators in various sectors is derived in appendix E.

Now are ready to define the highest weight representations in all sectors. The highest weight state is annihilated by positive modes of all generators:

$$\mathcal{O}_n \big| \, \mathrm{hws} \, \big\rangle = 0 \, , \qquad n > 0 \, . \tag{5.1}$$

To deal with the zero modes one should discuss the sectors separately.

5.1 NS sector

There are 3 zero modes: L_0, M_0, W_0. It is convenient to choose the highest weight state to be the eigenstate of these 3 operators and to label the highest weight representation by the correspondent eigenvalues. This is possible if the set of zero modes is commutative. One finds from (A.5) that the commutator

$$[M_0, W_0] = \frac{9\,\mu}{2\,c}(M_0 + {:}GH{:}_0) \tag{5.2}$$

is not zero. We rewrite the commutator by expanding ${:}GH{:}_0$ in the modes of G and H:

$$[M_0, W_0] = \frac{9\,\mu}{2\,c} \sum_{r=1/2}^{\infty} (G_{-r}\,H_r - H_{-r}\,G_r)\,. \tag{5.3}$$

The action of righthand side of (5.3) on highest weight state vanishes. This is what one effectively needs in order to choose the highest weight state to be the eigenstate of both M_0 and W_0. We define the notion of "effective" commutator: the commutation relation, which is true modulo terms, their action on highest weight state is zero. Concluding: the "effective" commutators of all three zero modes vanish, and one can label the highest weight representation by three weights, the eigenvalues of the zero modes:

$$\begin{aligned} L_0 \left| \, h\,, m\,, w \,\right\rangle &= h \left| \, h\,, m\,, w \,\right\rangle, \\ M_0 \left| \, h\,, m\,, w \,\right\rangle &= m \left| \, h\,, m\,, w \,\right\rangle, \\ W_0 \left| \, h\,, m\,, w \,\right\rangle &= w \left| \, h\,, m\,, w \,\right\rangle. \end{aligned} \tag{5.4}$$

One gets all states in the representation acting by negative modes on the highest weight state.

5.2 Ramond sector

There are 6 zero modes: L_0, M_0, W_0, G_0, H_0, U_0. Since L_0 is commutative with all other zero modes, it can be represented by a number h. The (anti)commutation relations of other zero modes are

$$\{G_0, G_0\} = 2\,(h - c/24)\,, \tag{5.5}$$

$$\{G_0, H_0\} = M_0\,, \tag{5.6}$$

$$[G_0, M_0] = \{G_0, U_0\} = 0\,, \tag{5.7}$$

$$[G_0, W_0] = U_0\,, \tag{5.8}$$

$$\{H_0, H_0\} = 2\,(h - c/24) + \lambda\, M_0 + 4/3\,\mu\, W_0\,, \tag{5.9}$$

$$[H_0, M_0] = 2/3\,\mu\, U_0\,, \tag{5.10}$$

$$[H_0, W_0] = \lambda/2\, U_0\,, \tag{5.11}$$

$$[M_0, W_0] = \{H_0, U_0\} = \frac{9\,\mu}{4\,c}\,(2\,G_0\,H_0 - M_0)\,, \tag{5.12}$$

$$[M_0, U_0] = \frac{9\,\mu}{2\,c}\,\left(2\,(h - c/24)\,H_0 - G_0\,M_0\right)\,, \tag{5.13}$$

$$[W_0, U_0] = \frac{1}{8c}\Big(54\,\mu\, U_0 - 54\,(h - c/24)\;(2\,G_0 + \lambda\, H_0) - \\ - 27\,\lambda\, G_0\, M_0 - 72\,\mu\, G_0\, W_0 + 54\, H_0\, M_0\Big)\,, \tag{5.14}$$

$$\{U_0, U_0\} = \frac{9}{4c}\Big(- 12\,(h - c/24)^2 - (h - c/24)\,(6\,\lambda\, M_0 + 8\,\mu\, W_0) + \\ + 4\,\mu\, G_0\, U_0 + 3\, M_0\, M_0\Big)\,. \tag{5.15}$$

They define finite dimensional $\mathcal{W}$ superalgebra. The commutation relations (5.5)–(5.11) are exact, and the commutators (5.12)–(5.15) are "effective", i.e. modulo terms which annihilate highest weight states.

The irreducible representations of the zero mode algebra are one-dimensional or two-dimensional and labeled by three weights: h, w and m. In one-dimensional representation the zero modes are given by

$$W_0 = w\,, \qquad M_0 = m\,, \qquad U_0 = 0\,, \\ G_0 = \sqrt{h - c/24}\,, \quad H_0 = \frac{m}{2\,\sqrt{h - c/24}}\,. \tag{5.16}$$

Such a representation exists only if the following condition is satisfied:

$$12\,(h - c/24)^2 + 6\,\lambda\,(h - c/24)\,m + 8\,\mu\,(h - c/24)\,w - 3\,m^2 = 0\,. \tag{5.17}$$

Taking $h = c/24$ one gets $G_0 = 0$. Such representation is called Ramond ground state. In the limit $h \to c/24$ the weight m approaches 0 like $m \sim (8\,(h - c/24)\,\mu\, w/3)^{1/2}$ and then $M_0 = 0$ and $H_0 = \sqrt{2\,\mu\, w/3}\,$.

The definition of the two-dimensional representations of the zero mode algebra (5.5)–(5.15) is more complicated. The bosonic zero modes L_0, M_0 and W_0 can not be diagonalized simultaneously, since M_0 and W_0 do not commute, even "effectively". One can label the highest weight representations by h, $w = \frac{1}{2}\mathrm{Trace}(W_0)$ and $m = \frac{1}{2}\mathrm{Trace}(M_0)$. In the following sections we will use another labels, but they will be always linearly dependent on the h, w, m. The maximal set of commuting operators contains 3 operators, which can be chosen as following: L_0, some fermionic operator F_0 and its square F_0^2. In section 7 it will be convenient to choose the F_0 operator as the zero mode of the $N = 1$ subalgebra supersymmetry generator.

5.3 First twisted sector

There are 3 zero modes: L_0, W_0 and H_0. The zero mode algebra is obtained by setting $\lambda = 0$ in (5.9) and (5.11):

$$\{H_0, H_0\} = 2\,(h - c/24) + 4/3\,\mu\, W_0\,, \\ [H_0, W_0] = 0\,. \tag{5.18}$$

Its irreducible representations are one dimensional. The highest weight state is labeled by two weights:

$$L_0\,|\, h\,, w\,\rangle = h|\, h\,, w\,\rangle\,, \\ W_0\,|\, h\,, w\,\rangle = w|\, h\,, w\,\rangle\,, \\ H_0\,|\, h\,, w\,\rangle = ((h - c/24) + 2/3\,\mu\, w)^{1/2}\,|\, h\,, w\,\rangle\,. \tag{5.19}$$

5.4 Second twisted sector

There are four zero modes: L_0, W_0, G_0 and U_0. Again L_0 is commutative with other zero modes and represented by its eigenvalue h. The commutation relations are

$$\begin{aligned}
\{G_0, G_0\} &= 2\,(h - c/24)\,, \\
[G_0, W_0] &= U_0\,, \\
\{G_0, U_0\} &= 0\,, \\
[W_0, U_0] &= \frac{1}{2\,c}\Big(27\,(9/48 - (h - c/24))\,G_0 + 9\,\mu\,(U_0 - 2\,G_0\,W_0)\Big)\,, \\
\{U_0, U_0\} &= \frac{1}{c}\Big(27\,(h - c/24)\,(9/48 - (h - c/24)) - \\
&\qquad - 18\,\mu\,(h - c/24)\,W_0 + 9\,\mu\,G_0\,U_0\Big)\,. \qquad (5.20)
\end{aligned}$$

The last two commutators are "effective".

Similarly to the Ramond sector the irreducible representations of the zero mode algebra are one or two-dimensional. They are labeled by two weights. In the one-dimensional representation the zero modes are

$$W_0 = w\,, \qquad G_0 = (h - c/24)^{1/2}\,, \qquad U_0 = 0\,. \qquad (5.21)$$

The representation exists only if

$$h - c/24 = 0 \qquad \text{or} \qquad 9 + 2\,c - 48\,h - 32\,\mu\,w = 0\,. \qquad (5.22)$$

The two-dimension representation is constructed in the following way. One can choose to diagonalize L_0 and W_0, and then the zero mode algebra is satisfied by

$$\begin{gathered}
W_0 = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \qquad G_0 = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}, \\
g = (h - c/24)^{1/2}, \qquad u = g\,(w_2 - w_1).
\end{gathered} \qquad (5.23)$$

There is a connection between w_1 and w_2:

$$2\,c\,(w_1 - w_2)^2 - 9\,\mu\,(w_1 + w_2) - 27\,(h - c/24) + 81/16 = 0\,. \qquad (5.24)$$

The two-dimensional highest weight state is labeled by two weights: h and w_1.

6. Minimal models

6.1 Unitary representations

As we have shown in section 4 the existence of $N{=}1$ superconformal subalgebras restricts the values of c and λ corresponding to unitary models of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra.

The unitary highest weight representations of an algebra are unitary with respect to all its real subalgebras. So there are also restrictions on the weights of unitary highest weight representations, coming from the non-unitarity theorem (appendix B) of the $N = 1$ superconformal algebra.

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In the region III of c, λ values (1.2), (1.3) there are 3 different $N=1$ superconformal subalgebras. The NS representation is labeled by three weights: h, w, m, which are linear functions of three weights $d^{(1)}$, $d^{(2)}$, $d^{(3)}$ of the three $N=1$ subalgebras of central charge $c^{N=1}_{k_1}$, $c^{N=1}_{k_2}$, $c^{N=1}_{k_1+k_2}$ respectively. The connection between the two sets of weights is taken from (4.8), (4.9), (4.10). The necessary condition for NS representation to be unitary is that the weights $d^{(1)}$, $d^{(2)}$, $d^{(3)}$ are included in the correspondent Kac tables (B.2) of conformal dimensions of the $N=1$ superconformal algebra. Therefore there is a finite number of unitary highest weight representations in the region III models, and we can call them the minimal models of the $\mathcal{SW}(3/2,3/2,2)$ algebra.

The highest weight representation of the $\mathcal{SW}(3/2,3/2,2)$ algebra can be decomposed to the sum of representations of the $N=1$ subalgebra. Let's take for example the third subalgebra ($c=c^{N=1}_{k_1+k_2}$). As we have shown in section 4 the generators of the $\mathcal{SW}(3/2,3/2,2)$ algebra fall to the $\Phi_{1,1}$ and $\Phi_{1,3}$ representations of the third subalgebra. The fusion rule for $\Phi_{1,3}$ is

$$\Phi_{1,3}\times\Phi_{m,n}=\Phi_{m,n-2}+\Phi_{m,n}+\Phi_{m,n+2}\,, \tag{6.1}$$

therefore the highest weight state of the $\mathcal{SW}(3/2,3/2,2)$ algebra with $d^{(3)}=d_{m,n}$ has descendants lying in the $\Phi_{m,n\pm2}$, $\Phi_{m,n\pm4}$, ...representation of the third subalgebra, and the $\mathcal{SW}(3/2,3/2,2)$ highest weight representation is decomposed to the sum of the third subalgebra representations with dimensions from the same row of the Kac table.

Applying these conclusions to the Ramond sector one gets that $\widetilde{T}^{(3)}_0$ in the two-dimensional representation looks like $\begin{pmatrix} d_{m,n} & 0\\ 0 & d_{m,n+2}\end{pmatrix}$ in the basis, where it is diagonal; $d_{m,n}$ is the dimension of Ramond representation of the $N=1$ superconformal algebra.

Although the zero modes of the three $N=1$ subalgebras $\widetilde{T}^{(1)}_0$, $\widetilde{T}^{(2)}_0$, $\widetilde{T}^{(3)}_0$, can not be diagonalized simultaneously, one still can label the Ramond representation by 3 pairs of Ramond dimensions (nearest Ramond neighbors in the correspondent Kac tables). Taking trace of zero modes in (4.8) one obtains the conformal dimension

$$\begin{aligned}h=\frac{1}{4}\Big(&-(d^{(3)}_{m_3,n_3}+d^{(3)}_{m_3,n_3+2})\,(k_1+k_2+2)+\\&+(d^{(1)}_{m_1,n_1}+d^{(1)}_{m_1+2,n_1})\,(k_1+4)+(d^{(2)}_{m_2,n_2}+d^{(2)}_{m_2+2,n_2})\,(k_2+4)\Big)\,.\end{aligned} \tag{6.2}$$

For the twisted sectors the situation is different. The twisted representations exist only in the minimal models with $k_1=k_2=k$. Since one cannot mix G and H generators in the twisted sector, the only $N=1$ subalgebra is the third subalgebra, its generators are given by (4.11) and (4.12). The two weights of the highest weight representation in the tw1 sector can be chosen to be the $\widetilde{T}^{(3)}_0$ eigenvalue $d^{(3)}$ and the conformal dimension h. The tw1 representation is of Ramond type with respect to the third $N=1$ subalgebra. The conditions (5.22) for existence of the one dimensional representation in the tw2 sector of the $k_1=k_2=k$ minimal model are rewritten as

$$h-c/24=0 \qquad \text{or} \qquad d^{(3)}=d_{k+2,k+2}\,. \tag{6.3}$$

The two-dimensional representation of the tw2 type is labeled by two weights: h and a pair $(d^{(3)}_1,d^{(3)}_2)$ of the nearest NS dimensions as the eigenvalues of the $\widetilde{T}^{(3)}_0$ operator. The

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connection (5.24), being rewritten in terms of $d_1^{(3)}$ and $d_2^{(3)}$, states exactly that $d_1^{(3)}$ and $d_2^{(3)}$ are nearest NS neighbors in the row of the correspondent Kac table.

All the conditions of unitarity so discussed are not sufficient. In the case $k_1 = 1$ one gets additional restrictions on unitary representations by noting that the energy momentum tensor can be decomposed to two commuting parts

$$T = \widetilde{T}^{(2)} + T^{\mathrm{Vir}}_{k_2+2} \qquad \text{or} \qquad T = \widetilde{T}^{(3)} + T^{\mathrm{Vir}}_{k_2}\,, \tag{6.4}$$

where T_k^{Vir} is the generator of the Virasoro algebra of central charge

$$c = c_k^{N=0} = 1 - \frac{6}{(k+2)(k+3)}\,, \tag{6.5}$$

corresponding to the unitary minimal models of the Virasoro algebra [18]. (One could see the decomposition from (3.5) and (3.6).) This fact restricts the values of $h - d^{(2)}$ and $h - d^{(3)}$. For the $k_1 = 1$ models all the discussed restrictions on the weights are in fact sufficient conditions of unitarity of NS and Ramond representations. (We do not prove it here.) The discussion for $k_1 = 1$ applies to the $k_2 = 1$ case as well.

By taking formally $k_1 = 0$ one should obtain the minimal models of $N=1$ superconformal algebra (appendix B).

In addition we know the examples (see section 6.3) of explicit construction of $k_1 = k_2 = 1$ and $k_1 = k_2 = 2$ minimal models of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra in terms of $N=2$ models.

Based on these facts and with the help of the coset construction (1.1) we guess the unitary spectrum of the general (arbitrary k_1 and k_2) $\mathcal{SW}(3/2, 3/2, 2)$ minimal model. The full list of minimal model unitary representations is presented in appendix D. The unitarity was also checked by computer calculations of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra Kac determinant on the few first levels.

The list of NS and Ramond representations forms a three-dimensional table with indices s_1, s_2, s_3, running in the range (D.1). The twisted sector representations form a two-dimensional table with indices t_1 and t_2 (D.8). There is a same number of NS and Ramond representations and the same number of tw1 and tw2 representations.

Substantial part of the spectrum could be predicted using the magic relation between the dimensions of any $N=1$ minimal model (B.2):

$$d^k_{m,1} + d^k_{1,n} - d^k_{m,n} = \frac{(m-1)(n-1)}{4} \qquad \forall k\,. \tag{6.6}$$

The relation is to be understood in the context of the fusion rule

$$\Phi_{m,1} \times \Phi_{1,n} = \Phi_{m,n}\,. \tag{6.7}$$

Taking $m = 3$ we get, that $\Phi_{1,n}$ fields are local or semilocal with respect to $\Phi_{3,1}$. The $\mathcal{SW}(3/2, 3/2, 2)$ algebra is decomposed to $\Phi_{1,1} \oplus \Phi_{3,1}$ representation of the first $N = 1$ subalgebra. Therefore the field $\Phi_{1,n}$ (of the first $N=1$ subalgebra) is a valid representation of the whole $\mathcal{SW}(3/2, 3/2, 2)$ algebra, since it is local (or semilocal) with respect to all the generators of $\mathcal{SW}(3/2,3/2,2)$. This representation is of Ramond or NS type, depending on n is even or odd respectively. The conformal dimension h of such a field coincides with the weight of the first $N=1$ subalgebra. For $n \leq 4$ the $\Phi_{1,n}$ field lies in the highest weight representation of the $\mathcal{SW}(3/2,3/2,2)$ algebra, for $n > 4$ it is descendant of some highest weight representation. We call such a representation the purely internal representation with respect to the first $N=1$ subalgebra. Obviously the set of purely internal representations is closed under fusion rules. (The similar situation is encountered in the case of the $\mathcal{SW}(3/2,2)$ algebra, which have purely internal representations with respect to its Virasoro subalgebra (section 4 of [5]).) Of course, there are also purely internal representations with respect to the second and to the third $N=1$ subalgebras. The representations $(s_1,1,1)$, $(1,s_2,1)$, $(1,1,s_3)$ are purely internal of the first, second and third subalgebras respectively. Such representations have simple fusion rules.

It is interesting to note, that $(1,s_2,s_3)$ fields of the $\mathcal{SW}(3/2,3/2,2)$ algebra are local or semilocal with respect to the $(3,1,1)$ field.

6.2 Fusion rules

It is well known that the fusion rules of NS and Ramond sector representations have $\mathbb{Z}_2$ grading: (NS $\to 0$, Ramond $\to 1$ under addition modulo 2). The $\mathcal{SW}(3/2,3/2,2)$ algebra has two additional twisted sectors: tw1 and tw2. The full set of fusion rules has $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading: NS $\to (0,0)$, Ramond $\to (1,0)$, tw1 $\to (0,1)$, tw2 $\to (1,1)$. The fusion rules of different sectors are summarized in table 1.

The fusions of the $\mathcal{SW}(3/2,3/2,2)$ representations have to be consistent with fusion rules of its subalgebras. In the case of minimal models there are three $N=1$ superconformal subalgebras and the fusions of the $\mathcal{SW}(3/2,3/2,2)$ NS and Ramond representations are completely fixed by the fusions of the correspondent $N=1$ minimal models (B.3). The fusion of (s_1',s_2',s_3') and (s_1'',s_2'',s_3'') representation (see appendix D) of the $\mathcal{SW}(3/2,3/2,2)$ algebra is

	NS	R	tw1	tw2
NS	NS	R	tw1	tw2
R	R	NS	tw2	tw1
tw1	tw1	tw2	NS	R
tw2	tw2	tw1	R	NS

Table 1: Sector fusion rules.

$$(s_1',s_2',s_3') \times (s_1'',s_2'',s_3'') = \sum_{s_1=|s_1'-s_1''|+1}^{\substack{\min(s_1'+s_1''-1,\\ 2k_1+3-(s_1'+s_1''))}} \; \sum_{s_2=|s_2'-s_2''|+1}^{\substack{\min(s_2'+s_2''-1,\\ 2k_2+3-(s_2'+s_2''))}} \; \sum_{s_3=|s_3'-s_3''|+1}^{\substack{\min(s_3'+s_3''-1,\\ 2(k_1+k_2)+7-(s_3'+s_3''))}} (s_1,s_2,s_3)\,, \tag{6.8}$$

where s_1, s_2 and s_3 are raised by steps of 2. The selection of one index is independent on two others and satisfies the "$su(2)$ pattern". The fusion rules of the $N=0$ and the $N=1$ minimal models satisfy the same pattern, the only difference is that in our case the table of representations is three-dimensional.

The proof is based on the first column of formula (D.3). The (s_1,s_2,s_3) representation is decomposed to the sum of representations of the first $N=1$ subalgebra from the column number s_1 of the correspondent $N=1$ Kac table. Thus the column selection rule of the first $N=1$ subalgebra is preserved and coincides with the s_1 selection in (6.8). Similarly the s_2 and s_3 selection rules are adopted from the column and row selection rules of the second and the third $N=1$ subalgebras respectively.

The twisted representations of the minimal models are labeled by two numbers: t_1 and t_2 (appendix D). But there is only one $N=1$ subalgebra (the third one) in the twisted sector. One can reed from (D.9) that t_1 is the row number in the correspondent Kac table, meaning that t_1 has common selection rules with s_3. The selection of t_2 in the fusion rules can not be fixed by the described methods.

The "corner" entries of the three-dimensional table of NS and Ramond representations are the $(1,1,1)$, $(k_1+1,1,1)$, $(1,k_2+1,1)$, $(1,1,k_1+k_2+3)$ representations. The first one is the vacuum representation. The three others have the following fusion "square":

$$\Phi \times \Phi = I\,, \tag{6.9}$$

where I denotes the identity (vacuum) representation. If such a field Φ is of the Ramond type, then the fusion of it with the other fields defines one-to-one transformation, mapping NS fields to Ramond ones. (This is an analogy of the $U(1)$ flow of the $N=2$ superconformal algebra [28].) If the field Φ is of the NS type, then its conformal dimension h is integer or half-integer and the $\mathcal{SW}(3/2,3/2,2)$ algebra can be extended to include this field. In the case, then both k_1 and k_2 are even, all three "corner" fields are of the NS type; in other cases (at least one k is odd) one "corner" fields is of the NS type and two are of the Ramond type (and then there are two different NS-R isomorphisms).

6.3 Examples

The following two examples are the $c=3/2$, $\lambda=0$ ($k_1=k_2=1$) and $c=9/4$, $\lambda=0$ ($k_1=k_2=2$) minimal models of the $\mathcal{SW}(3/2,3/2,2)$ algebra. The former model is realized as Z_2 orbifold of a tensor product of free fermion and free boson on radius $\sqrt{3}$, the latter is realized as Z_2 orbifold of the sixth ($c=9/4$) minimal model of the $N=2$ superconformal algebra.

6.3.1 $c=3/2,\ \lambda=0$ model

The boson on radius $\sqrt{3}$ is equivalent to the first minimal model of the $N=2$ superconformal algebra. The generators of $\mathcal{SW}(3/2,3/2,2)$ are constructed in the following way:

$$\begin{aligned} T &= T^{N=2}+T^{\rm Ising}\,, & M &= -\mathrm{i}\,3/\sqrt{2}\,G_2\,\psi\,,\\ G &= \sqrt{3}\,J\,\psi & W &= 1/\sqrt{2}\\ H &= \sqrt{3/2}\,G_1\,, & U &= \sqrt{2/3}\,(2\,J\,\partial\psi-\partial J\,\psi)\,, \end{aligned} \tag{6.10}$$

where $T^{N=2}$, G_1, G_2 and J are the (real) generators of the $N=2$ superconformal algebra at $c=1$, ψ is the free fermion field and $T^{\rm Ising}$ is its energy-momentum operator. The expressions in (6.10) are invariant under $\mathbb{Z}_2$ transformation $J\to -J, \psi\to-\psi$. One can build all the highest weight representations in the model (5 in NS, 5 in Ramond, 3 in every twisted sector) by appropriate combinations of representations of the $N=2$ minimal model and the Ising model. The list of representations is presented in table 3 in appendix D.

6.3.2 $c = 9/4$, $\lambda = 0$ model

The NS sector of the sixth ($c = 9/4$) $N = 2$ minimal model contains highest weight state Ψ_6^0 of conformal dimension 3/2 and zero $U(1)$ charge. The $N = 2$ superconformal algebra can be extended by $N = 2$ superprimary field corresponding to this state [29]. Then the fields, invariant under $\mathbb{Z}_2$ transformation $J \to -J$ form the $\mathcal{SW}(3/2, 3/2, 2)$ algebra:

$$\begin{aligned} T &= T^{N=2}\,, & H &= \Psi_6^0\,, & W &= (2T - 3\,{:}JJ{:})/\sqrt{5}\,, \\ G &= G_1\,, & M &= \Phi_6^0\,, & U &= 2/\sqrt{5}\,(3\,\mathrm{i}\,{:}JG_2{:} - \partial G_1)\,, \end{aligned} \tag{6.11}$$

where again $T^{N=2}$, G_1, G_2 and J are the (real) generators of the $N = 2$ superconformal algebra and $\big|\,\Phi_6^0\,\big\rangle$ is a superpartner of $\big|\,\Psi_6^0\,\big\rangle$: $\big|\,\Phi_6^0\,\big\rangle = (G_1)_{-\frac{1}{2}}\big|\,\Psi_6^0\,\big\rangle$. The highest weight representations of the $N = 2$ minimal model can be easily transformed to representations of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra. The list of highest weight representations consists of 16 NS, 16 Ramond, 6 tw1 and 6 tw2 representations (too long to be reproduced here explicitly).

7. Spectrum of the G_2 conformal algebra

In this section we discuss the $\mathcal{SW}(3/2, 3/2, 2)$ algebra at $c = 10\frac{1}{2}, \lambda = 0$. As we have shown in section 4 there is one real $N = 1$ subalgebra. It has central charge $\widetilde{c} = 7/10$ and thus coincides with the tricritical Ising model. The subalgebra is generated by operators $\widetilde{G}$ (4.11) and $\widetilde{T}$ (4.12). The $\mathcal{SW}(3/2, 3/2, 2)$ algebra is decomposed to the $\Phi_{1,1} \oplus \Phi_{3,1}$ representation of the $N = 1$ subalgebra. Since at $\widetilde{c} = 7/10$ the fields $\Phi_{3,1} = \Phi_{2,2}$ are identical there is a new null state on level 2:

$$3\,\widetilde{G}_{-3/2}\,\widetilde{G}_{-1/2} - 2\,\widetilde{T}_{-2}\,\big|\,\Phi_{3,1}\,\big\rangle\,. \tag{7.1}$$

(The null state on level 3/2 is already encoded in the structure of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra, but the existence of the null state (7.1) is a special feature of the tricritical Ising model.) Translating (7.1) to the language of generators of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra one gets that the null field is

$$2\sqrt{14}\,{:}GW{:} - 3\,{:}HM{:} + 2\,{:}TG{:} - 2\sqrt{14}\,\partial U\,. \tag{7.2}$$

This is an ideal of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra at $c = 10\frac{1}{2}, \lambda = 0$. (The existence of the ideal is known since [4].)

In ref. [1] the conformal algebra associated to the manifolds of G_2 holonomy is derived. Up to the ideal (7.2) the G_2 conformal algebra coincides with the $\mathcal{SW}(3/2, 3/2, 2)$ algebra at $c = 10\frac{1}{2}, \lambda = 0$. In the free field representation, used by the authors of [1] to obtain the G_2 algebra, the ideal (7.2) vanishes identically. The authors of ref. [1] used different basis of generators of the algebra. Their basis is connected to ours by

$$\begin{aligned} \Phi &= \mathrm{i}\,H\,, & X &= -(T + \sqrt{14}\,W)/3\,, \\ K &= \mathrm{i}\,M\,, & \tilde{M} &= -(\partial G + 2\sqrt{14}\,U)/6\,. \end{aligned} \tag{7.3}$$

The T and G generators are the same. We have explicitly checked, that the OPEs in the first appendix of [1] coincide (up to the ideal (7.2)) with the OPEs of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra.

Some unitary highest weight representations of the G_2 algebra are found in [1]. In this section we complete the list of unitary representations. Our calculation is based on the fact, that the T, G, W, U fields of the G_2 algebra generate closed subalgebra modulo the same ideal (7.2) and its descendants. This subalgebra is the $\mathcal{SW}(3/2, 2)$ superconformal algebra [12]–[17] and [5] of central charge $10\frac{1}{2}$. The G_2 algebra can be seen as an extended version of the $\mathcal{SW}(3/2, 2)$ algebra. It is interesting to note that the $\mathcal{SW}(3/2, 2)$ algebra at another value of central charge ($c = 12$) is the superconformal algebra associated to manifolds of Spin(7) holonomy [1].

NS		Ramond	
h	a	h	a
0	0	$7/16$	0
$3/8$	$3/80$	$7/16$	$3/80$
$1/2$	$1/10$	$7/16$	$1/10$
$7/8$	$7/16$	$7/16$	$7/16$
1	$3/5$	$15/16$	$(1/10, 3/5)$
$3/2$	$3/2$	$31/16$	$(3/5, 3/2)$
x	0	$7/16 + x$	$(0, 1/10)$
$3/8 + x$	$3/80$	$7/16 + x$	$3/80$
$1/2 + x$	$1/10$	$7/16 + x$	$(3/80, 7/16)$
$1 + x$	$3/5$	$15/16 + x$	$(1/10, 3/5)$

Table 2: The $c = 10\frac{1}{2}$ unitary model of the $\mathcal{SW}(3/2, 2)$ algebra.

The complete spectrum of unitary representations of the $\mathcal{SW}(3/2, 2)$ algebra is found in [5]. The $c = 10\frac{1}{2}$ unitary model spectrum is presented in table 2, where x stands for real positive number in the continuous spectrum representations. There are NS and Ramond representations, they are labeled by two numbers: the conformal dimension h and the internal dimension a (the weight of the $c = 7/10$ Virasoro subalgebra of the $\mathcal{SW}(3/2, 2)$ algebra).

The H field of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra is identified with the $h = 3/2$, $a = 3/2$ NS representation of the $\mathcal{SW}(3/2, 2)$ algebra, M is its superpartner.

The (H, M) supermultiplet is purely internal with respect to the $c = 7/10$ Virasoro subalgebra of the $\mathcal{SW}(3/2, 2)$ algebra. Due to this fact we know its fusion rules with all other representations of the $\mathcal{SW}(3/2, 2)$ algebra. In order to get the representations of $\mathcal{SW}(3/2, 3/2, 2)$ one have to combine the table 2 representations in multiplets under the action of the $h = 3/2, a = 3/2$ field. It can be easily done, however this way one gets only two weights. But we know that the $\mathcal{SW}(3/2, 3/2, 2)$ NS and Ramond representations are labeled by three weights. It is convenient to choose them: the conformal dimension h; the weight of the $N = 1$ subalgebra, which coincides with the $\mathcal{SW}(3/2, 2)$ algebra internal dimension a; and the eigenvalue m of the M zero mode. (In the case of two-dimensional Ramond representation m stands for $\frac{1}{2}\mathrm{Trace}(M_0)$.) The twisted representations are labeled by two weights: h and a.

One obtains the third weight m by using again the null field (7.2). Acting by it on the highest weight state one should get zero. Consider, for example, the NS sector. Take the $G_{-1/2}$ descendant of the ideal:

$$2\sqrt{14}\left(2{:}TW{:} - {:}GU{:} - \partial^2 W\right) - {:}G\partial G{:} + 3{:}H\partial H{:} - 3{:}MM{:} + 4{:}TT{:} + \partial^2 T\,. \tag{7.4}$$

The eigenvalue of its zero mode should be set to zero for a consistent representation. This leads to a connection between m and two other weights h and a:

$$m^2 = 10\,a\,(2\,h - 1)\,. \tag{7.5}$$

The similar connection can be found in the Ramond case. Concluding, the unitary representations of the G_2 algebra are (again $x > 0$):
NS sector:

$$\begin{aligned}
&1)\ h = a = m = 0\,,\\
&2)\ h = 1/2\,, \quad a = 1/10\,, \quad m = 0\,,\\
&3)\ h = x\,, \quad a = 0\,, \quad m = 0\,,\\
&4)\ h = 1/2 + x\,, \quad a = 1/10\,, \quad m = \sqrt{x}.
\end{aligned} \tag{7.6}$$

Ramond sector:

$$\begin{aligned}
&1)\ h = 7/16\,, \quad a = 7/16\,, \quad m = 0\,,\\
&2)\ h = 7/16\,, \quad a = 3/80\,, \quad m = 0\,,\\
&3)\ h = 7/16 + x\,, \quad a = (3/80, 7/16)\,, \quad m = 0\,,\\
&4)\ h = 7/16 + x\,, \quad a = 3/80, \quad m = \sqrt{x}/2.
\end{aligned} \tag{7.7}$$

tw1 sector:

$$\begin{aligned}
&1)\ h = 3/8\,, \quad a = 3/80\,,\\
&2)\ h = 7/8\,, \quad a = 7/16\,,\\
&3)\ h = 3/8 + x\,, \quad a = 3/80.
\end{aligned} \tag{7.8}$$

tw2 sector:

$$\begin{aligned}
&1)\ h = 7/16\,, \quad a = 1/10\,,\\
&2)\ h = 7/16\,, \quad a = 0\,,\\
&3)\ h = 7/16 + x\,, \quad a = (0, 1/10).
\end{aligned} \tag{7.9}$$

The NS and Ramond discrete spectrum states (the first two in (7.6) and (7.7)) were found in [1].

The first Ramond representation ($h = 7/16, a = 7/16, m = 0$) is purely internal with respect to the tricritical Ising model. Due to this fact we know its fusions. Since its square is identity, the field serves as an isomorphism mapping, connecting NS and Ramond sectors, and connecting the tw1 and tw2 sectors. The representations of the same line number in (7.6) and (7.7) and in (7.8) and (7.9) are isomorphic.

8. Summary

In this paper we study the $\mathcal{SW}(3/2, 3/2, 2)$ superconformal algebra. We show by explicit construction that the coset (1.1) contains the $\mathcal{SW}(3/2, 3/2, 2)$ algebra. The space of parameters (c and λ) is divided to two regions (figure 1).

In the first region there are unitary models at discrete points in the c, λ space. These are the minimal models of the algebra, they are described by the coset (1.1). In this region the $\mathcal{SW}(3/2, 3/2, 2)$ algebra has three different nontrivial $N=1$ subalgebras. The conformal dimensions with respect to these subalgebras serve as the weights of the $\mathcal{SW}(3/2, 3/2, 2)$ highest weight representations. The fusion rules are also dictated by the fusions of the $N = 1$ subalgebras. The characters of highest weight representations are not discussed in this paper. We suppose that the characters can be easily obtained from the coset construction.

In the second region of parameters the $\mathcal{SW}(3/2, 3/2, 2)$ algebra has one $N=1$ superconformal subalgebra. The unitary models "lie" on unitary curves and have continuous spectrum of unitary representations. One of the continuous unitary models is the $c = 10\frac{1}{2}$, $\lambda = 0$ model. The $\mathcal{SW}(3/2, 3/2, 2)$ algebra at these values of parameters coincides (up to a null field) with the superconformal algebra, associated to the manifolds of G_2 holonomy. From the other point of view it is an extended version of the $\mathcal{SW}(3/2, 2)$ algebra, which at another value of the central charge ($c = 12$) corresponds to the manifolds of Spin(7) holonomy. We find the unitary spectrum of the G_2 holonomy algebra. The connection of various realizations of the G_2 superconformal algebra with the geometric properties of the G_2 manifolds is the open problem for study.

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A. OPEs of the $\mathcal{SW}(3/2, 3/2, 2)$ algebra

T and G generate $N=1$ superconformal algebra of central charge c. (H, M) and (W, U) are its superprimary fields. The nontrivial OPEs are:

$$H(z)\,H(w) = \frac{\frac{2\,c}{3}}{(z-w)^3} + \frac{\lambda\,M + 2\,T + \frac{4\,\mu}{3}\,W}{z-w}\,, \tag{A.1}$$

$$H(z)\,M(w) = \frac{3\,G + 3\,\lambda\,H}{(z-w)^2} + \frac{-\frac{2\,\mu}{3}\,U + \partial G + \lambda\,\partial H}{z-w}\,, \tag{A.2}$$

$$M(z)\,M(w) = \frac{2\,c}{(z-w)^4} + \frac{4\,\lambda\,M + 8\,T + \frac{4\,\mu}{3}\,W}{(z-w)^2} + \frac{2\,\lambda\,\partial M + 4\,\partial T + \frac{2\,\mu}{3}\,\partial W}{z-w}\,, \tag{A.3}$$

$$H(z)\,W(w) = \frac{\mu\,H}{(z-w)^2} + \frac{\frac{\lambda}{2}\,U + \frac{\mu}{3}\,\partial H}{z-w}\,, \tag{A.4}$$

$$M(z)\,W(w) = \frac{\frac{\mu}{3}\,M + 2\,\lambda\,W}{(z-w)^2} + \frac{\frac{9\,\mu}{2\,c}\,{:}GH{:} + \frac{\mu\,(-27+2\,c)}{12\,c}\,\partial M + \lambda\,\partial W}{z-w}\,, \tag{A.5}$$

$$H(z)\,U(w) = \frac{\frac{-2\,\mu}{3}\,M + 2\,\lambda\,W}{(z-w)^2} + \frac{\frac{9\,\mu}{2\,c}\,{:}GH{:} - \frac{\mu\,(27+2\,c)}{12\,c}\,\partial M + \frac{\lambda}{2}\,\partial W}{z-w}\,, \tag{A.6}$$

$$\begin{aligned} M(z)\,U(w) = {}& \frac{2\,\mu\,H}{(z-w)^3} + \frac{\frac{5\,\lambda}{2}\,U + \frac{2\,\mu}{3}\,\partial H}{(z-w)^2} + \\ & + \frac{-\frac{9\,\mu}{2\,c}\,{:}GM{:} + \frac{9\,\mu}{c}\,{:}TH{:} + \lambda\,\partial U + \frac{\mu\,(-27+2\,c)}{12\,c}\,\partial^2 H}{z-w}\,, \end{aligned} \tag{A.7}$$

$$W(z)\,W(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T + \frac{\lambda}{2}\,M + \frac{(10\,c-27)\,\mu}{6\,c}\,W}{(z-w)^2} + \frac{\partial T + \frac{\lambda}{4}\,\partial M + \frac{(10\,c-27)\,\mu}{12\,c}\,\partial W}{z-w}\,, \tag{A.8}$$

$$\begin{aligned} W(z)\,U(w) = {}& \frac{-3\,G - \frac{3\,\lambda}{2}\,H}{(z-w)^3} + \frac{\frac{\mu\,(-27+10\,c)}{12\,c}\,U - \partial G - \frac{\lambda}{2}\,\partial H}{(z-w)^2} - \frac{1}{48\,c\,(z-w)} \times \\ & \times \Big(162\,\lambda\,{:}GM{:} + 432\,\mu\,{:}GW{:} - 324\,{:}HM{:} + 648\,{:}TG{:} + 324\,\lambda\,{:}TH{:} - \\ & \quad - 8\,\mu\,(27+2\,c)\,\partial U + 6\,(-27+2\,c)\,\partial^2 G + 3\,(-27+2\,c)\,\lambda\,\partial^2 H\Big)\,, \end{aligned} \tag{A.9}$$

$$U(z)\,U(w) = -\frac{2\,c}{(z-w)^5} - \frac{\frac{5\,\lambda}{2}\,M + 10\,T + \frac{\mu\,(-27+10\,c)}{3\,c}\,W}{(z-w)^3} -$$

$$-\frac{\frac{5\,\lambda}{4}\,\partial M+5\,\partial T+\frac{\mu\,(-27+10\,c)}{6\,c}\,\partial W}{(z-w)^2}-\frac{1}{16\,c\,(z-w)}\times$$
$$\times\Big(-144\,\mu\,{:}GU{:}-108\,{:}G\partial G{:}-54\,\lambda\,{:}G\partial H{:}+108\,{:}H\partial H{:}-108\,{:}MM{:}+$$
$$+216\,\lambda\,{:}TM{:}+432\,{:}TT{:}+288\,\mu\,{:}TW{:}+54\,\lambda\,{:}\partial GH{:}-$$
$$-3\,(9-2\,c)\,\lambda\,\partial^2 M+24\,c\,\partial^2 T-4\,\mu\,(27-2\,c)\,\partial^2 W\Big)\,. \tag{A.10}$$

where

$$\mu=\sqrt{\frac{9\,c\,(4+\lambda^2)}{2\,(27-2\,c)}} \tag{A.11}$$

and the fields in the right hand sides of the OPEs are taken in the point w.

B. Unitary minimal models of the $N=1$ superconformal algebra

At $c<3/2$ all unitary representations of the $N=1$ superconformal algebra are described by its minimal models. Their central charge is

$$c_k^{N=1}=\frac{3}{2}-\frac{12}{(k+2)(k+4)}\,,\qquad k=0,1,2,\ldots \tag{B.1}$$

The conformal dimensions of the unitary highest weight representations $\Phi_{m,n}$ of the $c=c_k^{N=1}$ minimal model are given in the Kac table

$$\begin{gathered}d^k_{m,n}=\frac{((k+2)\;m-(k+4)\;n)^2-4}{8\;(k+2)\;(k+4)}+\frac{r}{16}\,,\qquad \begin{aligned}m&=1,2,\ldots,k+3\,,\\ n&=1,2,\ldots,k+1\,,\end{aligned}\\ r=(m+n)\bmod 2=\begin{cases}0, & \text{NS sector,}\\ 1, & \text{Ramond sector.}\end{cases}\end{gathered} \tag{B.2}$$

The fusion rules are given by $su(2)$ like selection rules for every index (m and n):

$$\Phi_{m_1,n_1}\times\Phi_{m_2,n_2}=\sum_{m=|m_1-m_2|+1}^{\substack{\min(m_1+m_2-1,\\ 2k+7-(m_1+m_2))}}\;\sum_{n=|n_1-n_2|+1}^{\substack{\min(n_1+n_2-1,\\ 2k+3-(n_1+n_2))}}\Phi_{m,n}\,, \tag{B.3}$$

where the indices m and n in the sums are raised by steps of 2.

The $N=1$ minimal models correspond to the diagonal coset construction [20]:

$$\frac{su(2)_k\oplus su(2)_2}{su(2)_{k+2}}\,. \tag{B.4}$$

C. Coset construction of the $\mathcal{SW}(3/2,3/2,2)$ algebra

We present the explicit construction of the $\mathcal{SW}(3/2,3/2,2)$ generators in terms of the coset (1.1) currents. For details and notations see section 3.2.

$$T=\frac{1}{k_1+k_2+4}\Bigg(-\frac{1}{2}\,(k_1+k_2)\,{:}\psi_i\partial\psi_i{:}-2\,{:}J_i^{(1)}J_i^{(2)}{:}+$$

$$
\begin{aligned}
&+\Big(\Big(\frac{k_2+2}{k_1+2}:J_i^{(1)}J_i^{(1)}:-2:J_i^{(1)}J_i^{(3)}:\Big)+(1\leftrightarrow 2)\Big)\Big)\,,\\
G\,=\,&\frac{\sqrt{2}}{((k_1+2)(k_2+2)(k_1+k_2+4))^{1/2}}\times\\
&\times\Big(\mathrm{i}\,(k_1-k_2):\psi_1\psi_2\psi_3:+\Big((k_2+2):J_i^{(1)}\psi_i:-(1\leftrightarrow 2)\Big)\Big)\,,\\
H\,=\,&-\frac{1}{(3\,k_1\,k_2\,(k_1+2)(k_2+2)(k_1+k_2+4)(k_1+k_2+6))^{1/2}}\times\\
&\times\Big(3\,\mathrm{i}\,k_1\,k_2\,(k_1+k_2+4):\psi_1\psi_2\psi_3:+\Big(2\,k_2\,(2\,k_1+k_2+6):J_i^{(1)}\psi_i:+(1\leftrightarrow 2)\Big)\Big)\,,\\
M\,=\,&\frac{\lambda_{k_1,k_2}}{4\,(2\,k_1+k_2+6)(k_1+2\,k_2+6)}\Big(6\,(k_1\,k_2-2\,k_1-2\,k_2-12):J_i^{(1)}J_i^{(2)}:-\\
&-3\,k_1\,k_2\,(k_1+k_2+4):J_i^{(3)}J_i^{(3)}:+\Big(\Big(-\frac{6\,k_2\,(k_2+2)(2\,k_1+k_2+6)}{k_1-k_2}:J_i^{(1)}J_i^{(1)}:+\\
&+\frac{3\,k_2\,(k_1+k_2+4)(3\,k_1\,k_2+10\,k_1+2\,k_2+12)}{k_1-k_2}:J_i^{(1)}J_i^{(3)}:\Big)+(1\leftrightarrow 2)\Big)\Big)\,,\\
W\,=\,&\left(\frac{c_{k_1,k_2}}{9\,k_1\,k_2\,(k_1+2)(k_2+2)(k_1+k_2+4)(k_1+k_2+6)}\right)^{1/2}\times\\
&\times\Big(\frac{1}{2}k_1\,k_2(k_1+k_2+4):J_i^{(3)}J_i^{(3)}:+2(k_1\,k_2+4\,k_1+4\,k_2+12):J_i^{(1)}J_i^{(2)}:+\\
&\quad+\Big(\big(-k_2(k_2+2):J_i^{(1)}J_i^{(1)}:-2\,k_2(k_1+k_2+4):J_i^{(1)}J_i^{(3)}:\big)+(1\leftrightarrow 2)\Big)\Big)\,,\\
U\,=\,&\left(\frac{2\,c_{k_1,k_2}}{9\,k_1\,k_2\,(k_1+k_2+6)}\right)^{1/2}\times\\
&\times\left(-6\,\mathrm{i}:\begin{vmatrix} J_1^{(1)} & J_2^{(1)} & J_3^{(1)}\\ J_1^{(2)} & J_2^{(2)} & J_3^{(2)}\\ \psi_1 & \psi_2 & \psi_3\end{vmatrix}:+\Big(\big(k_2:\partial J_i^{(1)}\psi_i:-2\,k_2:J_i^{(1)}\partial\psi_i:\big)+(1\leftrightarrow 2)\Big)\right).
\end{aligned}
\tag{C.1}
$$

D. $\mathcal{SW}(3/2,3/2,2)$ minimal models

Here we present the complete list of unitary highest weight representations of the $\mathcal{SW}(3/2, 3/2,2)$ minimal models. The central charge c and the coupling λ of the (k_1,k_2) minimal model are given in (1.2) and (1.3). The list of NS and Ramond sector representations can be presented in the form of three-dimensional table with indices s_1, s_2, s_3:

$$
\begin{aligned}
s_1 &= 1,2,\ldots,k_1+1\,,\\
s_2 &= 1,2,\ldots,k_2+1\,,\\
s_3 &= 1,2,\ldots,k_1+k_2+3\,.
\end{aligned}
\tag{D.1}
$$

The representation (s_1,s_2,s_3) is of NS or Ramond type depending on $s_1+s_2+s_3$ is odd or even respectively. The highest weight representation is labeled by 3 weights $d^{(1)}$, $d^{(2)}$,

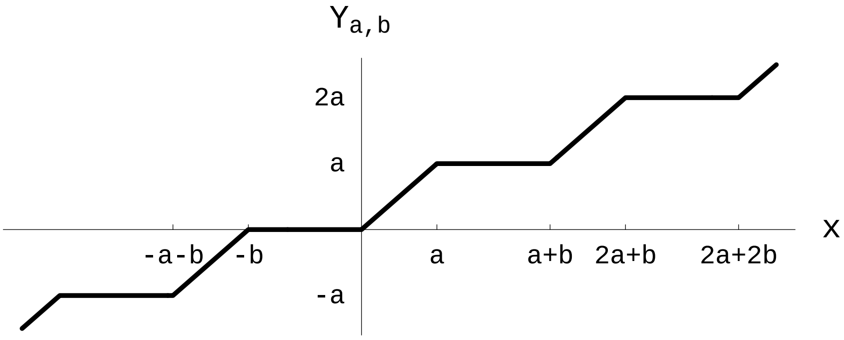

Figure 2: Definition of the $Y_{a,b}(x)$ function.

$d^{(3)}$, the conformal dimensions with respect to the three $N=1$ superconformal subalgebras. Their values are taken from the correspondent $N=1$ Kac tables (B.2):

$$d^{(1)} = d^{k_1}_{m_1,n_1}\,, \qquad d^{(2)} = d^{k_2}_{m_2,n_2}\,, \qquad d^{(3)} = d^{k_1+k_2}_{m_3,n_3}\,, \tag{D.2}$$

where the indices are connected to s_1, s_2, s_3 by

$$\begin{aligned} n_1 &= s_1\,, & m_1 &= s_1 - Y_1 + Y_2 + Y_3 \pm r\,,\\ n_2 &= s_2\,, & m_2 &= s_2 + Y_1 - Y_2 + Y_3 \pm r\,,\\ m_3 &= s_3\,, & n_3 &= s_3 + Y_1 + Y_2 - Y_3 \pm r\,. \end{aligned} \tag{D.3}$$

Y_1, Y_2, Y_3 are values of the $Y_{a,b}(x)$ function:

$$\begin{aligned} Y_1 &= Y_{2,2k_2+2}\,(s_1 - s_2 - s_3 + 1)\,,\\ Y_2 &= Y_{2,2k_1+2}\,(s_2 - s_3 - s_1 + 1)\,,\\ Y_3 &= Y_{2,2\min(k_1,k_2)+2}\,(s_3 - s_1 - s_2 + 1)\,. \end{aligned} \tag{D.4}$$

We define the function $Y_{a,b}(x)$ by its graph in figure 2.

The number r in (D.3) can be 0 or 1. It is 0 in the NS sector ($s_1 + s_2 + s_3$ odd). In the Ramond sector it is given by

$$\begin{aligned} r = 1 - \operatorname{sgn}\Big(&Y'_{2,2k_2+2}\,(s_1 - s_2 - s_3 + 1) + \\ &+ Y'_{2,2k_1+2}\,(s_2 - s_3 - s_1 + 1) + Y'_{2,2\min(k_1,k_2)+2}\,(s_3 - s_1 - s_2 + 1)\Big)\,, \end{aligned} \tag{D.5}$$

where $Y'_{a,b}(x)$ is a derivative of $Y_{a,b}(x)$ with respect to x. The Y' function is not continuous, but the values in the points of discontinuity are not important. r distinguishes between one ($r = 0$) and two-dimensional ($r = 1$) Ramond representations.

The conformal dimension h (the eigenvalue of L_0 operator) is calculated from $d^{(1)}$, $d^{(2)}$, $d^{(3)}$ weights by

$$h = \frac{1}{2}\Big(- d^{(3)}\,(k_1 + k_2 + 2) + d^{(1)}\,(k_1 + 4) + d^{(2)}\,(k_2 + 4)\Big) \tag{D.6}$$

in the case of one-dimensional (NS or Ramond) representation. In the case of two-dimensional Ramond representation the $d^{(1)}$, $d^{(2)}$, $d^{(3)}$ in (D.6) should be substituted by half of the sum of the correspondent $N\!=\!1$ Ramond dimensions (6.2).

The following representations are identical:

$$(s_1, s_2, s_3) = (k_1 + 2 - s_1, k_2 + 2 - s_2, k_1 + k_2 + 4 - s_3)\,. \tag{D.7}$$

There are $[\frac{(k_1+1)(k_2+1)(k_1+k_2+3)+1}{4}]$ NS representations and the same number of Ramond representations.

The $k_1 = k_2 = k$ minimal models contain two additional twisted sectors: tw1 and tw2. The list of tw1 and tw2 representations can be arranged in the two-dimensional table with indices t_1 and t_2:

$$\begin{aligned} t_1 &= 1, 2, \ldots, k+2\,, \\ t_2 &= 1, 2, \ldots, k+1\,. \end{aligned} \tag{D.8}$$

The $t_1 + t_2$ even entries are of tw2 type and the $t_1 + t_2$ odd entries are of tw1 type. The representations in the twisted sectors are labeled by the conformal dimension h and the weight of the third $N\!=\!1$ subalgebra $d^{(3)}$:

$$\begin{aligned} & h = \frac{|t_1 - t_2|}{4} + \frac{t_2^2 - t_1^2 + \delta}{8\,(k+2)}\,, && \delta = \begin{cases} k-1, & \text{tw1}, \\ 3\,k/2, & \text{tw2}; \end{cases} \\ & d^{(3)} = d^{(2\,k)}_{m,n}\,, \quad \begin{aligned} m &= t_1, \\ n &= t_1 + \operatorname{sgn}(t_2 - t_1) \pm r, \end{aligned} && r = \begin{cases} 0, & \text{tw1}, \\ \operatorname{sgn}(t_2 - t_1), & \text{tw2}. \end{cases} \end{aligned} \tag{D.9}$$

Again r distinguishes between one ($r = 0$) and two-dimensional ($r = 1$) tw2 representations. There are $(k\!+\!1)(k\!+\!2)/2$ tw1 representations and the same number of tw2 representations.

We want to illustrate the formulas of the present appendix by some explicit examples. The simplest model is the $k_1 = k_2 = 1$ ($c = 3/2, \lambda = 0$) model, discussed in section 6.3.1. Its highest weight representations are presented in table 3. We use the $\begin{pmatrix} h \\ d^{(1)}, d^{(2)}, d^{(3)} \end{pmatrix}$ notation for the NS and Ramond representations and the $\begin{pmatrix} h \\ d^{(3)} \end{pmatrix}$ notation for the twisted sectors. One of the four NS/Ramond weights is dependent on other three and is presented for convenience only. The Ramond and tw1 sectors are *slanted*. The $h = 9/16$ Ramond representation is two-dimensional.

The second example is the $k_1 = 2, k_2 = 3$ ($c = 37/15, \lambda = -182/(405\sqrt{11})$) minimal model. Since the list of representations is too long, we reproduce only the conformal dimensions h of the highest weight representations (table 4).

E. Mode expansions of normal ordered products

Here we derive the formula for the mode expansion of normal ordered product of operators in various sectors. The normal ordered product $:\!PQ\!:$ is defined as the zero order term in

$s_1 = 1$

$s_3 \downarrow$ \ $s_2 \rightarrow$		
	$\begin{pmatrix} 0 \\ 0,0,0 \end{pmatrix}$	$\begin{pmatrix} \frac{7}{16} \\ \frac{3}{80}, \frac{7}{16}, \frac{3}{8} \end{pmatrix}$
	$\begin{pmatrix} \frac{1}{16} \\ \frac{3}{80}, \frac{3}{80}, \frac{1}{16} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{8} \\ 0, \frac{1}{10}, \frac{1}{16} \end{pmatrix}$
	$\begin{pmatrix} \frac{1}{6} \\ \frac{1}{10}, \frac{1}{10}, \frac{1}{6} \end{pmatrix}$	$\begin{pmatrix} \frac{5}{48} \\ \frac{3}{80}, \frac{3}{80}, \frac{1}{24} \end{pmatrix}$
	$\begin{pmatrix} \frac{9}{16} \\ \begin{smallmatrix} \frac{3}{80} \\ \frac{7}{16} \end{smallmatrix}, \begin{smallmatrix} \frac{3}{80} \\ \frac{7}{16} \end{smallmatrix}, \begin{smallmatrix} \frac{1}{16} \\ \frac{9}{16} \end{smallmatrix} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{8} \\ \frac{1}{10}, 0, \frac{1}{16} \end{pmatrix}$
	$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{10}, \frac{1}{10}, 0 \end{pmatrix}$	$\begin{pmatrix} \frac{7}{16} \\ \frac{7}{16}, \frac{3}{80}, \frac{3}{8} \end{pmatrix}$

$s_1 = 2$

$s_3 \downarrow$ \ $s_2 \rightarrow$		
	$\begin{pmatrix} \frac{7}{16} \\ \frac{7}{16}, \frac{3}{80}, \frac{3}{8} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{10}, \frac{1}{10}, 0 \end{pmatrix}$
	$\begin{pmatrix} \frac{1}{8} \\ \frac{1}{10}, 0, \frac{1}{16} \end{pmatrix}$	$\begin{pmatrix} \frac{9}{16} \\ \begin{smallmatrix} \frac{3}{80} \\ \frac{7}{16} \end{smallmatrix}, \begin{smallmatrix} \frac{3}{80} \\ \frac{7}{16} \end{smallmatrix}, \begin{smallmatrix} \frac{1}{16} \\ \frac{9}{16} \end{smallmatrix} \end{pmatrix}$
	$\begin{pmatrix} \frac{5}{48} \\ \frac{3}{80}, \frac{3}{80}, \frac{1}{24} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{6} \\ \frac{1}{10}, \frac{1}{10}, \frac{1}{6} \end{pmatrix}$
	$\begin{pmatrix} \frac{1}{8} \\ 0, \frac{1}{10}, \frac{1}{16} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{16} \\ \frac{3}{80}, \frac{3}{80}, \frac{1}{16} \end{pmatrix}$
	$\begin{pmatrix} \frac{7}{16} \\ \frac{3}{80}, \frac{7}{16}, \frac{3}{8} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0,0,0 \end{pmatrix}$

twisted

$t_1 \downarrow$ \ $t_2 \rightarrow$		
	$\begin{pmatrix} \frac{1}{16} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{3}{8} \\ \frac{3}{8} \end{pmatrix}$
	$\begin{pmatrix} \frac{1}{8} \\ \frac{1}{16} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{16} \\ \frac{1}{16} \end{pmatrix}$
	$\begin{pmatrix} \frac{11}{48} \\ \frac{1}{6} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{24} \\ \frac{1}{24} \end{pmatrix}$

Table 3: The $k_1 = k_2 = 1$ $(c = 3/2, \lambda = 0)$ minimal model.

$s_1 = 1$

$s_3 \downarrow$ \ $s_2 \rightarrow$				
	0	$\frac{27}{80}$	$\frac{9}{10}$	$\frac{31}{16}$
	$\frac{5}{48}$	$\frac{1}{15}$	$\frac{121}{240}$	$\frac{7}{6}$
	$\frac{5}{18}$	$\frac{83}{720}$	$\frac{8}{45}$	$\frac{103}{144}$
	$\frac{37}{48}$	$\frac{7}{30}$	$\frac{41}{240}$	$\frac{1}{3}$
	$\frac{5}{6}$	$\frac{161}{240}$	$\frac{7}{30}$	$\frac{13}{48}$
	$\frac{175}{144}$	$\frac{61}{90}$	$\frac{443}{720}$	$\frac{5}{18}$
	$\frac{7}{6}$	$\frac{241}{240}$	$\frac{17}{30}$	$\frac{29}{48}$
	$\frac{23}{16}$	$\frac{9}{10}$	$\frac{67}{80}$	1

$s_1 = 2$

$s_3 \downarrow$ \ $s_2 \rightarrow$				
	$\frac{3}{8}$	$\frac{27}{80}$	$\frac{31}{40}$	$\frac{23}{16}$
	$\frac{5}{48}$	$\frac{53}{120}$	$\frac{121}{240}$	$\frac{25}{24}$
	$\frac{11}{72}$	$\frac{83}{720}$	$\frac{199}{360}$	$\frac{103}{144}$
	$\frac{13}{48}$	$\frac{13}{120}$	$\frac{41}{240}$	$\frac{17}{24}$
	$\frac{17}{24}$	$\frac{41}{240}$	$\frac{13}{120}$	$\frac{13}{48}$
	$\frac{103}{144}$	$\frac{199}{360}$	$\frac{83}{720}$	$\frac{11}{72}$
	$\frac{25}{24}$	$\frac{121}{240}$	$\frac{53}{120}$	$\frac{5}{48}$
	$\frac{23}{16}$	$\frac{31}{40}$	$\frac{27}{80}$	$\frac{3}{8}$

$s_1 = 3$

$s_3 \downarrow$ \ $s_2 \rightarrow$				
	1	$\frac{67}{80}$	$\frac{9}{10}$	$\frac{23}{16}$
	$\frac{29}{48}$	$\frac{17}{30}$	$\frac{241}{240}$	$\frac{7}{6}$
	$\frac{5}{18}$	$\frac{443}{720}$	$\frac{61}{90}$	$\frac{175}{144}$
	$\frac{13}{48}$	$\frac{7}{30}$	$\frac{161}{240}$	$\frac{5}{6}$
	$\frac{1}{3}$	$\frac{41}{240}$	$\frac{7}{30}$	$\frac{37}{48}$
	$\frac{103}{144}$	$\frac{8}{45}$	$\frac{83}{720}$	$\frac{5}{18}$
	$\frac{7}{6}$	$\frac{121}{240}$	$\frac{1}{15}$	$\frac{5}{48}$
	$\frac{31}{16}$	$\frac{9}{10}$	$\frac{27}{80}$	0

Table 4: The $k_1 = 2, k_2 = 3$ minimal model.

OPE:

$$P(z)\, Q(w) = \sum_{k=1}^{N} \frac{R^{(k)}(w)}{(z-w)^k} + {:}PQ{:}(w) + O(z-w)\,. \tag{E.1}$$

The well known formula for the mode expansion of $:PQ:$

$$:PQ:_n = \sum_{m \le -\Delta_P} P_m Q_{n-m} + (-1)^{PQ} \sum_{m \ge -\Delta_P+1} Q_{n-m} P_m \,, \tag{E.2}$$

is valid only if m has the same modding as Δ_P, i.e. m runs on integer or half integer numbers depending on the spin of P is integer or half integer respectively. So in the NS sector the expansion (E.2) works. In the case of Ramond or twisted sectors the formula should be modified.

The idea of the following calculation is taken from ref. [30, (section 3)] , where the mode expansion of $:G^+G^-:$ was obtained using the same method. (G^+ and G^- are the supersymmetry generators of the $N = 2$ superconformal algebra.) Let's calculate the integral:

$$\oint_0 \mathrm{d}w \, w^{n+\Delta_P+\Delta_Q-1} \oint_w \mathrm{d}z \, \frac{1}{z-w} \, z^{\epsilon} \, P(z) \, Q(w) \, w^{-\epsilon} \,, \tag{E.3}$$

where the first integration is around w and the second is around 0. The integral (E.3) is equal to the n-mode

$$:PQ:_n + \sum_{k=1}^{N} \binom{\epsilon}{k} R_n^{(k)} \,. \tag{E.4}$$

The integration contour in (E.3) can be transformed to

$$\oint\oint_{z>w} \mathrm{d}z \, \mathrm{d}w - \oint\oint_{w>z} \mathrm{d}w \, \mathrm{d}z \,. \tag{E.5}$$

The z^{ϵ} term in the integration function was introduced to compensate the phase change of $P(z)$ around $z = 0$.

Expanding $(z-w)^{-1}$ and integrating one gets

$$:PQ:_n = -\sum_{k=1}^{N} \binom{\epsilon}{k} R_n^{(k)} + \sum_{m \le -\Delta_P+\epsilon} P_m Q_{n-m} + (-1)^{PQ} \sum_{m \ge -\Delta_P+1+\epsilon} Q_{n-m} P_m \,, \tag{E.6}$$

where m runs on $\mathbb{Z} - \Delta_P + \epsilon$. ϵ is usually chosen to be 0 or 1/2 to produce the correct modding for operator P. In the case $\epsilon = 0$ we get back the formula (E.2) as expected. Note that the formula (E.6) is valid for any ϵ (not only 0 or 1/2) consistent with the chosen modding.

Another approach to the calculation of mode expansions of composite operators is presented in [16, (section 3)] and in [5, (appendix C)] and leads to the same results.

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